CCFU Proof 1

φ is the unique positive attracting fixed point of C_2

Given. The second-order linear recurrence with unit coefficients:

$$C_2: \quad x_{n+1} = x_n + x_{n-1}.$$

Define. $R_n = x_{n+1}/x_n$, assuming $x_n \neq 0$ and $R_0 > 0$.

Step 1 — Ratio dynamics.

$$R_{n+1} = \frac{x_{n+2}}{x_{n+1}} = \frac{x_{n+1} + x_n}{x_{n+1}} = 1 + \frac{1}{R_n}.$$

Step 2 — Fixed point. At equilibrium $R_{n+1} = R_n = R$:

$$R = 1 + \frac{1}{R}, \qquad R^2 = R + 1, \qquad R^2 - R - 1 = 0.$$

Solutions: $R = (1 \pm \sqrt{5})/2$. The unique positive root is

$$\varphi = \frac{1 + \sqrt{5}}{2}.$$

Step 3 — Local stability. Define $f(R) = 1 + 1/R$. Then $f'(R) = -1/R^2$, so

$$|f'(\varphi)| = \frac{1}{\varphi^2} = \frac{1}{\varphi + 1} < 1,$$

where we used $\varphi^2 = \varphi + 1$. The fixed point is locally attracting.

Step 4 — Global convergence. For all $R_0 > 0$:

$$R_{n+1} - \varphi = 1 + \frac{1}{R_n} - \varphi = \frac{\varphi - R_n}{\varphi\, R_n} = -\frac{R_n - \varphi}{\varphi\, R_n}.$$

Therefore

$$|R_{n+1} - \varphi| = \frac{|R_n - \varphi|}{\varphi\, R_n}.$$

For any $R_0 > 0$, we have $R_1 = 1 + 1/R_0 > 1$. Therefore for all $n \geq 1$:

$$R_n > 1, \qquad \varphi\, R_n > \varphi > 1,$$

so

$$|R_{n+1} - \varphi| < \frac{|R_n - \varphi|}{\varphi}.$$

The error contracts by a factor smaller than $1/\varphi$ per step. Convergence is geometric after the first iteration.

Therefore $R_n \to \varphi$ for all $R_0 > 0$.

Conclusion. $\varphi = (1+\sqrt{5})/2$ is the unique positive fixed point of the ratio map $f(R) = 1+1/R$. It is globally attracting on $\mathbb{R}_+$. No parameters are chosen. The recurrence C_2 alone forces φ. ■